COMPUTING BOUNDS FOR FUNCTIONAL OUTPUTS OF EXACT SOLUTIONS OF PDE'S

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We develop techniques for computing bounds for functional outputs of the exact solution of partial differential equations. Existing techniques for approximating the solutions of PDE's rely on the experience of the user to estimate a-priori the size of the discretization required to resolve the various problem features. Failure to appropriately do so, may result in results which are either too expensive to obtain or worse yet, uncertain. Modern a-posteriori and mesh adaptivity methods have the potential to alleviate this problem but not to eliminate it completely, i.e., a saturation hypothesis needs to be made that can not be verified a-priori.

Consider for instance the functional

$$l(v) : X \rightarrow I\!R \quad ,$$

where X is an infinite dimensional space which includes the solution, u, of a; partial differential equation

$$\mathcal{A}\, u = 0 \ .$$

Thus, we are interested in the particular output $s = l(u)$. Traditionally, one computes a finite dimensional approximation $u_h \in X_h$, (usually $X^h \subset X$) to the PDE, and then, evaluates $s_h = l(u_h)$.

There exists an extensive body of knowledge regarding the approximation properties of various numerical schemes. In principle, one can compute approximate solutions which are arbitrarily close to the exact solution according to some measure, but in general, there is no algorithm that will deliver the exact solution in a finite number of operations.

Our objective is to try to determine upper and lower bounds, s^+ and S^-, which are strict. We shall see that, in order to accomplish this, we bypass the step of computing an approximation u_h, to the infinite dimensional solution u, and attempt to determine the bounds for the scalar $s = l(u)$, directly. Our approach is completely hands-off and delivers bounds for the outputs of interest for certain classes of partial differential equations. With this approach, certainty is guaranteed and is independent of the size of the computation. Increased computational cost, on the other hand, means that the upper and lower bounds produced are closer (i.e., smaller bound gap). When the equations considered are those of linear elasticity and the relevant output is the total energy, our approach reduces tot he well known complementary energy method [1]. However, our method generalizes to arbitrary linear outputs of the solution and to non-symmetric equations for which a variational method does not exist.

The work that will be presented is carried out in collaboration with colleagues at MIT, the University of Wales at Swansea, and the Universitat Politecnica de Barcelona.

References

[1] B. Fraeijs de Veubeke *Displacement and equilibrium models in the finite element method*, in B.M. Fraeijs de Veubeke Memorial Volume of Selected Papers, M. Geradin, ed., Sijthof and Noordhoff International Publishers, 1980.